MIXED METHODS FOR DEGENERATE ELLIPTIC PROBLEMS AND APPLICATION TO FRACTIONAL LAPLACIAN

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In this work we analyze the approximation by mixed finite element methods of degenerate second order elliptic problems.

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz polytope and ω be a non-negative measurable function. We assume that the boundary is decomposed into two disjoint parts Γ_D and Γ_N. Given $g \in L^2(\Omega)$ and $f \in L^2(\Gamma_N)$ we consider the problem

$$\begin{cases} -\mathrm{div}\,(\omega \nabla u) = g & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ -\omega \nabla u \cdot \boldsymbol{n} = f & \text{on } \Gamma_N \end{cases} \tag{1}$$

where $\boldsymbol{n}$ denotes the unit exterior normal vector. If $\Gamma_N = \partial\Omega$ we assume the usual compatibility condition $\int_\Omega g = \int_{\partial\Omega} f$.

We are interested in degenerate problems in the sense that the coefficient ω can become infinite or zero in subsets of $\overline{\Omega}$ with vanishing n-dimensional measure.

Introducing the vector field variable $\boldsymbol{\sigma} = -\omega\nabla u$, problem (1) can be transformed into the equivalent first order system

$$\begin{cases} \boldsymbol{\sigma} + \omega \nabla u = 0 & \text{in } \Omega \\ \mathrm{div}\,\boldsymbol{\sigma} = g & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \boldsymbol{\sigma} \cdot \boldsymbol{n} = f & \text{on } \Gamma_N \end{cases}$$

Then, mixed finite element methods are based on a weak formulation of this system and they approximate simultaneously $\boldsymbol{\sigma}$ and u.

As an application of our results we will consider a problem arising in the solution of the fractional Laplace equation.